#3 Applications of Derivatives

1. *Analysis of Curves*

I. Increasing and Decreasing (3.3)

- Let f be a continuous and differentiable function on a given interval

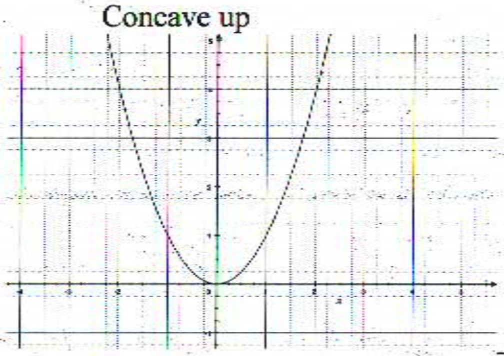
A function is increasing on an interval if $f'(x)$ is positive for all x on that interval
A function is decreasing on an interval if $f'(x)$ is negative for all x on that interval
A function is constant on an interval if $f'(x)$ is zero for all x on that interval

II. Concavity (3.4)

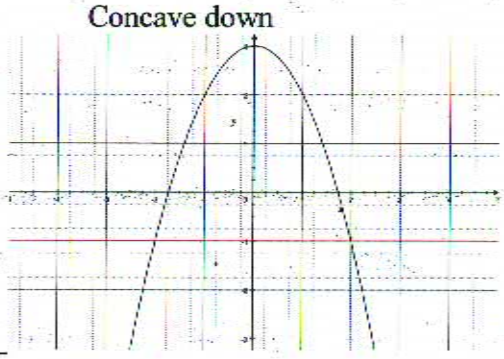
- a function is concave up if it's graph opens upwards
  - *hint: could hold water
- a function is concave down if it's graph opens downwards
  - *hint: could not hold water

- Testing for Concavity
  - a function is concave up on an interval if $f''(x)$ is positive for all x in that interval
  - a function is concave down on an interval if $f''(x)$ is negative for all x in that interval
  - we know nothing about a function if $f''(x)$ is zero

Concave up



Concave down



Example:
Determine the open intervals on which the graph of $f(x) = 6/(x^2 + 3)$ is concave upward and downward.

Solution:

1) First make sure that the function is continuous on the entire real line
2) Find the second derivative of the function
   a. Rewrite the original function $f(x) = 6(x^2 + 3)^{-1}$
   b. Differentiate $f(x) = -12x/(x^2 + 3)^2$
   c. Differentiate again $f(x) = 36(x^2-1)/ (x^2 + 3)^3$
3) Set the second derivative equal to zero to find inflection points
   $x = -1$ and $+1$

4) Test f''(x) for the intervals (- infinity, -1), (-1,1), and (1, infinity) and make a table showing the results.

Interval	Infinity < x < -1	-1 < x < 1	1 < x < infinity
Test Value	x = -2	x = 0	x = 2
Sign of f''(x)	f''(-2) > 0	f''(0) < 0	f''(2) >0
Conclusion	Concave upward	Concave downward	Concave upward

III. Monotonicity (9.1)

- a function is <u>monotonic</u> if its terms are all nondecreasing or nonincreasing
- a function is strictly monotonic if its terms are all increasing or all decreasing

Example

Determine whether each sequence having the given nth term is monotonic

a. $a_n = 3 + (-1)^n$ **b.** $b_n = 2n/(1 + n)$ **c.** $n^2/(2^n - 1)$

Solution:

a. This sequence alternates between 2 and 4. Therefore it is not monotonic.

b. This sequence is monotonic because each term is greater than the one before it. This can be seen if one compares the terms b_n to b_{n-1}.

c. This sequence is not monotonic because the second term is greater than both the first and third terms. However, if the first term was ignored, this sequence would be monotonic.

2. *Planar Curves*

I. Parametric form (10.2-10.3)

A) Definition

- a <u>planar curve</u> is a pair of parametric equations and their graph

$x = f(t)$ - where t = parameter

$y = g(t)$

B) Derivatives in parametric form

- the slope (velocity) at (x, y) is

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right)$$

- the second derivative (acceleration at (x, y) is

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example
For the curve given by

$$x=\sqrt{t} \qquad \text{and} \qquad y=\frac{1}{4}(t^2-4)$$

find the slope and concavity at the point (2, 3)

Solution:
1) find the first derivative in parametric form

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{\left(\frac{1}{2}\right)t}{\left(\frac{1}{2}\right)^{\frac{-1}{2}}}=t^{\frac{3}{2}}$$

2) Find the second derivative in parametric form

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{\left(\frac{3}{2}\right)t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)t^{\frac{-1}{2}}}=3t$$

3) Using the original equations, we see that t = 4 at (2,3)
4) Therefore the slope is $(4)^{3/2} = 8$
5) When t = 4, the second derivative is 3(4) = 12 > 0
Therefore, the graph is concave upward at (2,3)

II. Polar form (10.4- 10.5)

A) Definition
- polar form uses an angle measurement and a radius length to plot points and graphs on the polar coordinate system

$$r=f(\theta)$$

B) Converting from rectangular coordinates to polar coordinates
1. $x = r\cos\theta$ $\qquad$ $y = r\sin\theta$
2. $\tan\theta = y/x$ $\qquad$ $r^2 = x^2 + y^2$

C) Derivatives in polar form
-the slope (velocity) is

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)=\frac{(f(\theta)\cos\theta+f'(\theta)\sin\theta)}{(-f(\theta)\sin\theta+f'(\theta)\cos\theta)}$$

Example

Find the horizontal and vertical tangent lines of $r = \sin\theta$, $0 \leq \theta \leq \pi$

Solution:

1. Write the equation in parametric form.
   $x = r\cos\theta = \sin\theta\cos\theta$
   $y = r\sin\theta = \sin^2\theta$
2. Differentiate x and y and set each equal to zero
   $dx/d\theta = \cos^2\theta - \sin^2\theta = \cos 2\theta = 0 \quad \Rightarrow \quad \theta = \pi/4, 3\pi/4$
   $dy/d\theta = 2\sin\theta\cos\theta = \sin 2\theta = 0 \quad \Rightarrow \quad \theta = 0, \pi/2$
3. Therefore, the graph has vertical tangent lines at $(\sqrt{2}/2, \pi/4)$ and $(\sqrt{2}/2, 3\pi/4)$ and horizontal tangent lines at $(0,0)$ and $(1,\pi/2)$

III. Vector Form (11.1-

A) Definitions

<u>scalar quantities</u>- quantities characterized by a single real number
<u>vector-</u> has both magnitude and direction

B) Component Form

- if **v** is a vector whose initial point is at the origin and whose terminal point is at $<v_1, v_2>$, then its <u>component form</u> is $\mathbf{v} = <v_1, v_2>$

C) Converting Directed Line Segments to Component Form

- if $P(p_1, p_2)$ and $Q(q_1, q_2)$ are initial and terminal points, then vector $<v_1, v_2> = <q_1 - p_1, q_2 - p_2>$

- the magnitude (length) of **v** is $||v|| = \sqrt{(q1\text{-}p1)^2 - (q2\text{-}p2)^2}$

D) Vector Addidtion and Scalar Multiplication

$\mathbf{u} + \mathbf{v} = <u_1 + v_1, u_2 + v_2>$
$c\mathbf{u} = <cu_1, cu_2>$ where c = scalar
$-\mathbf{v} = <-v_1, -v_2>$

E) The Unit Vector

If **v** is a nonzero vector, then its unit vector is

$$u = \frac{1}{||v||}v$$

F) Stadard Unit Vectors

Standard unit vectors can be used to represent any vector

$\mathbf{i} = <1,0>$ **i** = horizontal component
$\mathbf{j} = <0,1>$ **j** = vertical component
$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$

Example
Find the component form and length of the vector **v** that has initial point (3, -7) and terminal point (-2,5)
Solution:

1. Let $P(3, -7) = (p_1, p_2)$ and $Q(q_1, q_2)$ and $\mathbf{v} = \langle v_1, v_2 \rangle$
2. $v_1 = q_1 - p_1 = -2 - 3 = -5$
   $v_2 = q_2 - p_2 = 5 - (-7) = 12$
   Therefore $\mathbf{v} = \langle -5, 12 \rangle$
3. $\|\mathbf{v}\| = \sqrt{((-5^2) + 12^2)} = \sqrt{169} = 13$

Example 2
Let **u** be the vector with initial point (2, -5) and terminal point (-1, 3), and let $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$. Write each vector as a linear combination of **i** and **j**.
a. u   **b.** $2\mathbf{u} - 3\mathbf{v}$

Solution
a. $\mathbf{u} = <q_1 - p_1, q_2 - p_2> = <-3,8> = -3\mathbf{i} + 8\mathbf{j}$

b. $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v} = 2(-3\mathbf{i} + 8\mathbf{j}) - 3(2\mathbf{i} - \mathbf{j}) = -12\mathbf{i} - 19\mathbf{j}$

3. *Optimization* (3.7)

I. Critical points

A) occur where $f'(x) = 0$ or is undefined
B) critical points are potential extrema
Note: if f is continuous on [a, b], then f has both a minimum and a maximum on the interval.

II. First Derivative Test for Extrema

-used to determine if a critical point at $x = c$ is an extrema

1) if $f'(x)$ changes from positive to negative at $x = c$, then c is a relative (local) maximum
2) if $f'(x)$ changes from negative to positive at $x = c$, then c is a relative (local) minimum
3) if $f'(x)$ doesn't change sign at $x = c$, there is no extrema

III. Optimization

- the process of finding a minimum or a maximum value as a solution to a problem.

1) Find an equation for the value you're trying to optimize (best if it is in terms of a single variable
2) Find critical points
3) Use critical points and end points (if applicable) to determine maximum and/or minimum value(s)

Example

Given a piece of sheet metal with identical squares cut out of the four corners, determine the size of the squares that yields the maximum volume for the box formed by folding up the sides.

1) Draw a Picture
2) Set up equations
   length = $(17 - 2x)$ width = $(11 - 2x)$ height = x
   $v(x) = 4x^3 - 56x^2 + 187x$
3) Find critical points:
   $v'(x) = 12x^2 - 112x + 187 = 0$ at $x = 7.156$ and 2.178
4) Check critical points with second derivative test:
   $v''(x) = 24x - 112$ $v''(7.156) = 59.74$ $v''(2.178) = -59.72$
5) Therefore the maximum volume is achieved at $x = 2.178$

Example 2

Find the length and width of a recagle with a perimeter of 100 meters and a maximum area.

Solution

1) Set up equation with one variable
   $A = l(w)$ $100 = 2l + 2w$ $w = 50 - l$
   $A = l(50 - l)$ $A = 50l - l^2$
2) Find critical points
   $A' = 50 - 2l = 0$ at $l = 25$
3) Check critical points with second derivative test
   $A'' = -2$
4) Therefore, the maximum area is achieved when the length is 25 meters and the width is 25 meters.

See textbook Ch 2.2

4. *Rate of Change*

A. **[Def]** the speed at which a variable changes over a specific period of time (answers.com)

B. **[Application]** one can use a derivative "to determine the rate of change of one variable with respect to another" (Larson 113)

C. **[Common Usage: Position]**

i. the motion of an object moving in a straight line

a. this is usually represented by a horizontal or vertical line

b. up & right is described as the "positive direction" while down & left is described as the "negative direction"

c. the function *s* (position) as a function of *t* (time) is called a position function

d. over the period time Δt the position change of an object is given by $\Delta s = s(t+\Delta t) - s(t)$, or rate=distance/time

D. **[Example]**

i. Find dy/dt when $x = 7$ for the equation $f(x) = x^2 - 3x + 2$ given that $dx/dt = 4$

$dy/dt = (2x - 3)(dx/dt)$ *Given:* $dx/dt = 4, x = 7$

$dy/dt = (2*7 - 3)(4) = 44$

E. **[Applied Contexts]**

i. Displacement of a Falling Object:

$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$ ➤ where g is acceleration due to gravity (-9.8 m/s or -32 ft/s on Earth)

ii. Velocity: one can use the derivative of the displacement to obtain velocity, given by:

$s'(t) = v(t) = gt + v_0$ for *instantaneous velocity*

$\Delta s/\Delta t = [s(t+\Delta t) - s(t)]/\Delta t$ for *average velocity* (change in distance/change in time)

*Note: speed is the absolute value of velocity

iii. Acceleration: one can use the derivative of velocity to obtain acceleration

F. **[Example]**

Joe throws a ball (straight down) from the top of a 550-foot building with an initial velocity of 17 feet per second.

i. Find the *average velocity* on the interval [1,3]

$s(t) = \frac{1}{2}(-32)t^2 - v_0t + s_0$

$s(t) = -16t^2 - 17t + 550$

$s(1) = -16(1)^2 - 17(1) + 550 = 517$ ft

$s(3) = -16(3)^2 - 17(3) + 550 = 355$ ft

$\Delta t = 3-1 = 2$ seconds

Average Velocity $= [s(3) - s(1)]/\Delta t = (355-517)/2 = -81$ ft/s

ii. Find the *instantaneous velocity* when t = 2

$v(t) = gt + v_0$

$v(t) = -32(2) + 17 = -47$ ft/s

iii. Find the time when the ball hits the ground.

$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$

$0 = \frac{1}{2}(-32)t^2 + 17t + 550$

$t = -5.356, 6.418$

Solution: 6.418 seconds

See textbook Ch 2.6

5. *Related Rates*

A. **[Def]** a physical quantity over time (wolframalpha.com)

B. **[Application]** related rates use the Chain Rule "to find the rate of change of two or more related variables that are changing with respect to *time*" (Larson 149)

C. **[The Process]** (as taken from Mr. Wagner's notes)

i. List out the given variables and rates, listed as derivatives, with the values

ii. Find an equation(s) that relates the different quantities
iii. Differentiate the equation implicitly
iv. Solve for the unknowns and answer the question

D. **[Examples]**

i. *An 8 foot long ladder is leaning against a wall. The top of the ladder is sliding down the wall at the rate of 2 feet per second. How fast is the bottom of the ladder moving along the ground at the point in time when the bottom of the ladder is 4 feet from the wall?*

[1] y = distance from the top of the ladder to the ground
x = distance from the bottom of the ladder to the wall
$dy/dt = -2$ Find dx/dt when $x = 4$ and $y = 4(3^{1/2})$ (by the Pythagorean Theorem)

[2] $x^2 + y^2 = 64$

[3] $2x\, dx/dt + 2y\, dy/dt = 0$
$2x\, dx/dt - 4y = 0$

[4] $dx/dt = 4y/(2x) = 2(3^{1/2})$ ft/sec when x = 4 ft

ii. *Water is being poured into a conical reservoir at the rate of ∏ cubic feet per second. The reservoir has a radius of 6 feet across the top and a height of 12 feet. At what rate is the depth of the water increasing when the depth is 6 feet?*

The volume of the water in the reservoir (V) is given by:

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \pi \ ft^3 / \sec \quad \text{(given)}$$

$$r = \frac{1}{2}h \quad \text{(using similar triangles)}$$

$$V = \frac{1}{12}\pi h^3$$

$$\pi = \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{h^2}$$

$$\frac{dh}{dt} = \frac{1}{9} ft / \sec \quad \text{when } h = 6 \ ft$$

See textbook Ch 5.3

6. *Implicit Differentiation and the Link to an Inverse's Derivative*

A. One can use implicit differentiation to find the derivative of an inverse function. This is especially useful when y is difficult to solve for.

B. **[Steps]**

i. Differentiate x with respect to y.
ii. Plug in the given values (if any).
iii. Find the reciprocal of dx/dy in order to get dy/dx.

B. **[Example]**

Find dy/dx at the given point for the equation.

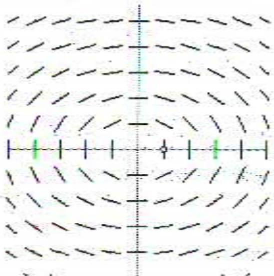
$x = y^3 - 7y^2 + 2$ (-4,1)

dx/dy = $3y^2 - 14y$ Differentiate.
dx/dy = $3(1)^2 - 14(1)$ Plug in given values.
dx/dy = -11, therefore, *dy/dx* = -1/11

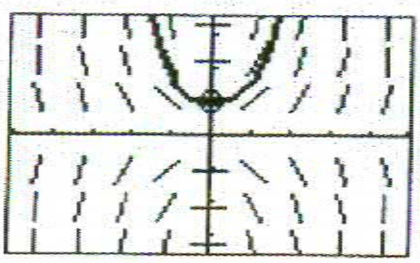
See textbook Ch 6.1/6.2

7. Differential equations, slope fields and solution curves

1. Definitions
   a. Differential equation – An equation involving one dependent variable and its derivatives with respect to one more independent variables.
      – An equation that involves the derivatives of a function as well as the function itself.
   b. Slope fields –
      i. Direction fields for a differential equation.
      ii. A graph of short line segments with slopes given by the differential equation. Each line segment has the same slope as the solution curve through that point.
   c. Solution curve – Curve(s) represented by the solution of a first-order differential equation.
      i. General Solution – a family of solutions to a differential equation. There is one solution curve for each value assigned to the arbitrary constant (C).
      ii. Particular Solution – single solution to a differential equation based on initial conditions. There is only one solution curve.
2. Explanation
   a. How to use a slope field – A slope field shows the general shape of all solutions to a differential equation. The line segments give a visual perspective of the directions of the solutions of the differential equation.
   b. How to draw a slope field for a differential equation – At each point on the graph, draw a tiny line segment passing through the point with the slope of the equation at that point.
   c. How to draw solution curves – Start at a point and "follow the flow lines" to come up with a curve. For a particular solution there will be a single curve while for a general solution there will be infinitely many solution curves (+C).
3. Examples
   a. Draw the slope field for this differential equation: dy/dx = x/y



   b. Draw the slope field for the differential equation dydx = xy and sketch the solution curve through the point (0, 1).



******Go to http://www.calculusapplets.com/slopefields.html and type in a differential equation to see the resulting slope field produced.

See textbook Ch 6.1

8. *Euler's Method to solve differential equations*

1. Definitions
   c. Differential equations – An equation that involves the derivatives of a function as well as the function itself.
   – An expression in which the derivative appears as a variable.

   a. Euler's Method – A numerical approach to a particular solution to a differential equation.
2. Explanation
   a. When to use Euler's Method – Use Euler's method if you have the derivative of a function, but cannot find the integral, or if you just want to approximate the value of a function at a given point.
   b. Euler's Method is a numerical approach to approximating the particular solution of $y' = F(x,y)$ that passes through the point (x_0, y_0). Start with the point (x_0, y_0) and a slope of $F(x_0, y_0)$. From there, $x_n = x_{n-1} + h$, $y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$
3. Example:

Given differential equation $y' = e^{xy}$ and starting point $y(0) = 1$; find $y(1)$ using $n = 5$ steps.

$y' = e^{xy}$ $y(0) = 1$ $y(1) =$

$n = 5$ steps $h = 0.2$

Draw table:

n	0	1	2	3	4	5
x_n	0					
y_n	1					

$y_n = y_{n-1} + hF(x, y)$ $F(x,y) = y'$

$y_1 = y_0 + 0.2e^{x_0 y_0} = 1 + 0.2 = 1.2$

$y_2 = 1.2 + 0.2e^{(.2)(1.2)} = 1.45$

y_3

n	0	1	2	3	4	5
x_n	0	.2	.4	.6	.8	1.0
y_n	1	1.2	1.45	1.721	2.064	2.303

$\boxed{y(1) = 2.303}$

See textbook Ch 8.7/8.8

9. *L'Hospital's Rule*

1. Definitions
   a. L'Hospital's Rule – Suppose that we have one of the following cases,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad \text{OR} \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. L'Hospital's Rule says that:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

b. Improper integral – A definite integral that has either or both limits infinite or an integrand that approaches infinity at one or more points in the range of integration.

2. Explanation

a. L'Hospital's Rule in 3 steps:

Step 1. Check that the limit of $\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. If it is not, then L'Hospital's Rule cannot be used (unless the limit is rewritten to match the form $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$)

Step 2. Differentiate f and g separately. [*Note*: Do not differentiate $\frac{f(x)}{g(x)}$ using the quotient rule!]

Step 3. Find the limit of $\frac{f'(x)}{g'(x)}$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $\frac{f(x)}{g(x)}$. If the limit is an indeterminate form of type $\frac{0}{0}$, then simplify $\frac{f'(x)}{g'(x)}$ algebraically and apply L'Hospital's Rule again.

3. Examples

(1) $\lim_{x\to 0}\frac{e^x-x-1}{x^2}=$

Use L'Hopital's Rule (twice): $\lim_{x\to 0}\frac{e^x-x-1}{x^2}=\lim_{x\to 0}\frac{e^x-1}{2x}=\lim_{x\to 0}\frac{e^x}{2}=\frac{1}{2}$

(2) $\lim_{x\to+\infty}\frac{x}{(\ln x)^3}=\frac{\infty}{\infty}$

Use L'Hopital's Rule (three times):

$$\lim_{x\to+\infty}\frac{x}{(\ln x)^3}=\lim_{x\to+\infty}\frac{1}{3(\ln x)^2\left(\frac{1}{x}\right)}=\lim_{x\to+\infty}\frac{x}{3(\ln x)^2}=\lim_{x\to+\infty}\frac{1}{6\ln x\left(\frac{1}{x}\right)}=\lim_{x\to+\infty}\frac{x}{6\ln x}=$$

$$\lim_{x\to+\infty}\frac{1}{6\left(\frac{1}{x}\right)}=\lim_{x\to+\infty}\frac{x}{6}=+\infty$$

(3) $\lim_{x\to0^+}\sqrt[3]{x}\ln x=0\bullet\infty$

Must be rewritten: $=\lim_{x\to0^-}\frac{\ln x}{x^{-1/3}}=$

Now use L'Hospital's Rule: $=\lim_{x\to0^-}\frac{\frac{1}{x}}{-\frac{1}{3}x^{-4/3}}=\lim_{x\to0^+}\frac{-3x^{4/3}}{x}=\lim_{x\to0^-}(-3\sqrt[3]{x})=0.$